THE PRINCIPLES

OF

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THE PRINCIPLES
OF
MATHEMATICS

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PREFACE.

THE present work has two main objects. One of these, the proof that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles, is undertaken in Parts II.—VII. of this Volume, and will be established by strict symbolic reasoning in Volume II. The demonstration of this thesis has, if I am not mistaken, all the certainty and precision of which mathematical demonstrations are capable. As the thesis is very recent among mathematicians, and is almost universally denied by philosophers, I have undertaken, in this volume, to defend its various parts, as occasion arose, against such adverse theories as appeared most widely held or most difficult to disprove. I have also endeavoured to present, in language as untechnical as possible, the more important stages in the deductions by which the thesis is established.

The other object of this work, which occupies Part I., is the explanation of the fundamental concepts which mathematics accepts as indefinable. This is a purely philosophical task, and I cannot flatter myself that I have done more than indicate a vast field of inquiry, and give a sample of the methods by which the inquiry may be conducted. The discussion of indefinables—which forms the chief part of philosophical logic—is the endeavour to see clearly, and to make others see clearly, the entities concerned, in order that the mind may have that kind of acquaintance with them which it has with redness or the taste of a pineapple. Where, as in the present case, the indefinables are obtained primarily as the necessary residue in a process of analysis, it is often easier to know that there must be such entities than actually to perceive them; there is a process analogous to that which resulted in the discovery of Neptune, with the difference that the final stage—the search with a mental telescope for the entity which has been inferred—is often the most difficult part of the undertaking. In the case of classes, I must confess, I have failed to perceive any concept fulfilling the conditions requisite for the notion of *class*. And the contradiction discussed in Chapter x. proves that something is amiss, but what this is I have hitherto failed to discover.

The second volume, in which I have had the great good fortune to secure the collaboration of Mr A. N. Whitehead, will be addressed exclusively to mathematicians; it will contain chains of deductions, from the premisses of symbolic logic through Arithmetic, finite and infinite, to Geometry, in an order similar to that adopted in the present volume; it will also contain various original developments, in which the method of Professor Peano, as supplemented by the Logic of Relations, has shown itself a powerful instrument of mathematical investigation.

The present volume, which may be regarded either as a commentary upon, or as an introduction to, the second volume, is addressed in equal measure to the philosopher and to the mathematician; but some parts will be more interesting to the one, others to the other. I should advise mathematicians, unless they are specially interested in Symbolic Logic, to begin with Part IV., and only refer to earlier parts as occasion arises. The following portions are more specially philosophical: Part I. (omitting Chapter II.); Part II., Chapters XI., XV., XVI., XVII.; Part III.; Part IV., § 207, Chapters XXVI., XXVII., XXXI.; Part V., Chapters XLI., XLII., XLIII.; Part VI., Chapters L., LI., LII.; Part VII., Chapters LIII., LIV., LV., LVII., LVIII.; and the two Appendices, which belong to Part I., and should be read in connection with it. Professor Frege's work, which largely anticipates my own, was for the most part unknown to me when the printing of the present work began; I had seen his *Grundgesetze der Arithmetik*, but, owing to the great difficulty of his symbolism, I had failed to grasp its importance or to understand its contents. The only method, at so late a stage, of doing justice to his work, was to devote an Appendix to it; and in some points the views contained in the Appendix differ from those in Chapter VI., especially in §§ 71, 73, 74. On questions discussed in these sections, I discovered errors after passing the sheets for the press; these errors, of which the chief are the denial of the null-class, and the identification of a term with the class whose only member it is, are rectified in the Appendices. The subjects treated are so difficult that I feel little confidence in my present opinions, and regard any conclusions which may be advocated as essentially hypotheses.

A few words as to the origin of the present work may serve to show the importance of the questions discussed. About six years ago, I began an investigation into the philosophy of Dynamics. I was met by the difficulty that, when a particle is subject to several forces, no one of the component accelerations actually occurs, but only the resultant acceleration, of which they are not parts; this fact rendered illusory such causation of particulars by particulars as is affirmed, at first sight, by the law of gravitation. It appeared also that the difficulty in regard to absolute motion is insoluble on a relational theory of space. From these two questions I was led to a re-examination of the principles of Geometry, thence to the philosophy of continuity and infinity, and thence, with a view to discovering the meaning of the word *any*, to Symbolic Logic. The final outcome, as regards the philosophy of Dynamics, is perhaps rather slender; the reason of this is, that almost all the problems of Dynamics appear to me empirical, and therefore outside the scope of such a work as the present. Many very interesting questions have had to be omitted, especially in Parts VI. and VII., as not relevant to my purpose, which, for fear of misunderstandings, it may be well to explain at this stage.

When actual objects are counted, or when Geometry and Dynamics are applied to actual space or actual matter, or when, in any other way, mathematical reasoning is applied to what exists, the reasoning employed has a form not dependent upon the objects to which it is applied being just those objects that they are, but only upon their having certain general properties. In pure mathematics, actual objects in the world of existence will never be in question, but only hypothetical objects having those general properties upon which depends whatever deduction is being considered; and these general properties will always be expressible in terms of the fundamental concepts which I have called logical constants. Thus when space or motion is spoken of in pure mathematics, it is not actual space or actual motion, as we know them in experience, that are spoken of, but any entity possessing those abstract general properties of space or motion that are employed in the reasonings of geometry or dynamics. The question whether these properties belong, as a matter of fact, to actual space or actual motion, is irrelevant to pure mathematics, and therefore to the present work, being, in my opinion, a purely empirical question, to be investigated in the laboratory or the observatory. Indirectly, it is true, the discussions connected with pure mathematics have a very important bearing upon such empirical questions, since mathematical space and motion are held by many, perhaps most, philosophers to be self-contradictory, and therefore necessarily different from actual space and motion, whereas, if the views advocated in the following pages be valid, no such self-contradictions are to be found in mathematical space and motion. But extra-mathematical considerations of this kind have been almost wholly excluded from the present work.

On fundamental questions of philosophy, my position, in all its chief features, is derived from Mr G. E. Moore. I have accepted from him the non-existential nature of propositions (except such as happen to assert existence) and their independence of any knowing mind; also the pluralism which regards the world, both that of existents and that of entities, as composed of an infinite number of mutually independent entities, with relations which are ultimate, and not reducible to adjectives of their terms or of the whole which these compose. Before learning these views from him, I found myself completely unable to construct any philosophy of arithmetic, whereas their acceptance brought about an immediate liberation from a large number of difficulties which I believe to be otherwise insuperable. The doctrines just mentioned are, in my opinion, quite indispensable to any even tolerably satisfactory philosophy of mathematics, as I hope the following pages will show. But I must leave it to my readers to judge how far the reasoning assumes these doctrines, and how far it supports them. Formally, my premisses are simply assumed; but the fact that they allow mathematics to be true, which most current philosophies do not, is surely a powerful argument in their favour.

In Mathematics, my chief obligations, as is indeed evident, are to Georg Cantor and Professor Peano. If I had become acquainted sooner with the work of Professor Frege, I should have owed a great deal to him, but as it is I arrived independently at many results which he had already established. At every stage of my work, I have been assisted more than I can express by the suggestions, the criticisms, and the generous encouragement of Mr A. N. Whitehead; he also has kindly read my proofs, and greatly improved the final expression of a very large number of passages. Many useful hints I owe also to Mr W. E. Johnson; and in the more philosophical parts of the book I owe much to Mr G. E. Moore besides the general position which underlies the whole.

In the endeavour to cover so wide a field, it has been impossible to acquire an exhaustive knowledge of the literature. There are doubtless many important works with which I am unacquainted; but where the labour of thinking and writing necessarily absorbs so much time, such ignorance, however regrettable, seems not wholly avoidable.

Many words will be found, in the course of discussion, to be defined in senses apparently departing widely from common usage. Such departures, I must ask the reader to believe, are never wanton, but have been made with great reluctance. In philosophical matters, they have been necessitated mainly by two causes. First, it often happens that two cognate notions are both to be considered, and that language has two names for the one, but none for the other. It is then highly convenient to distinguish between the two names commonly used as synonyms, keeping one for the usual, the other for the hitherto nameless sense. The other cause arises from philosophical disagreement with received views. Where two qualities are commonly supposed inseparably conjoined, but are here regarded as separable, the name which has applied to their combination will usually have to be restricted to one or other. For example, propositions are commonly regarded as (1) true or false, (2) mental. Holding, as I do, that what is true or false is not in general mental, I require a name for the true or false as such, and this name can scarcely be other than *proposition*. In such a case, the departure from usage is in no degree arbitrary. As regards mathematical terms, the necessity for establishing the existence-theorem in each case—*i.e.* the proof that there are entities of the kind in question—has led to many definitions which appear widely different from the notions usually attached to the terms in question. Instances of this are the definitions of cardinal, ordinal and complex numbers. In the two former of these, and in many other cases, the definition as a class, derived from the principle of abstraction, is mainly recommended by the fact that it leaves no doubt as to the existence-theorem. But in many instances of such apparent departure from usage, it may be doubted whether more has been done than to give precision to a notion which had hitherto been more or less vague.

For publishing a work containing so many unsolved difficulties, my apology is, that investigation revealed no near prospect of adequately resolving the contradiction discussed in Chapter x., or of acquiring a better insight into the nature of classes. The repeated discovery of errors in solutions which for a time had satisfied me caused these problems to appear such as would have been only concealed by any seemingly satisfactory theories which a slightly longer reflection might have produced; it seemed better, therefore, merely to state the difficulties, than to wait until I had become persuaded of the truth of some almost certainly erroneous doctrine.

My thanks are due to the Syndics of the University Press, and to their Secretary, Mr R. T. Wright, for their kindness and courtesy in regard to the present volume.

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